NCERT Solutions Class 11 Maths Chapter 8 Binomial Theorem

Question 1:

Expand the expression $(1-2x)^5$

Solution:

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as

$$\begin{aligned}(1-2x)^5 &= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)(2x)^4 - {}^5C_5(2x)^5 \\ &= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5) \\ &= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5\end{aligned}$$

Question 2:

Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Solution:

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$\begin{aligned}\left(\frac{2}{x}-\frac{x}{2}\right)^5 &= {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5 \\ &= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32} \\ &= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}\end{aligned}$$

Question 3:

Expand the expression $(2x-3)^6$

Solution:

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$$\begin{aligned}(2x-3)^6 &= {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6 \\ &= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 729 \\ &= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729\end{aligned}$$

Question 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Solution:

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 - {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$

$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

Question 5:

Expand the expression $\left(x+\frac{1}{x}\right)^6$

Solution:

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \left(\frac{1}{x^6}\right)$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Solution:

96 can be expressed as the difference of two numbers whose powers are easier to calculate.
Hence, $96 = 100 - 4$
Therefore,

$$(96)^3 = (100-4)^3$$

$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$$

$$= (100)^3 - 3(10000)(4) + 3(100)(16) - (64)$$

$$= 1000000 - 120000 + 4800 - 64$$

$$= 1004800 - 120064$$

$$= 884736$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Solution:

102 can be expressed as the sum of two numbers whose powers are easier to calculate.

Hence, $102 = 100 + 2$

Therefore;

$$\begin{aligned}(102)^5 &= (100+2)^5 \\ &= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5 \\ &= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32 \\ &= 11040808032\end{aligned}$$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Solution:

101 can be expressed as the sum of two numbers whose powers are easier to calculate.

Hence $101 = 100 + 1$

Therefore,

$$\begin{aligned}(101)^4 &= (100+1)^4 \\ &= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4 \\ &= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4 \\ &= 100000000 + 4000000 + 60000 + 400 + 1 \\ &= 104060401\end{aligned}$$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

Solution:

99 can be expressed as the difference of two numbers whose powers are easier to calculate.

Hence, $99 = 100 - 1$

Therefore,

$$\begin{aligned}(99)^5 &= (100-1)^5 \\ &= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5 \\ &= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1 \\ &= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1 \\ &= 10010000500 - 500100001 \\ &= 9509900499\end{aligned}$$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ be obtained as

$$\begin{aligned}(1.1)^{10000} &= (1+0.1)^{10000} \\ &= {}^{10000}C_0 + {}^{10000}C_1(0.1) + \textit{other positive terms} \\ &= 1 + 10000\times(0.1) + \textit{other positive terms} \\ &= 1 + 1000 + \textit{other positive terms} \\ &= 1001 + \textit{other positive terms} \\ &> 1000\end{aligned}$$

Hence, $(1.1)^{10000} > 1000$.

Question 11:

Find $(a+b)^4 - (a-b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

Solution:

Using Binomial Theorem, the expression $(a+b)^4$ and $(a-b)^4$ can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$

Therefore,

$$(a+b)^4-(a-b)^4=\left[\begin{array}{r}\left({}^4C_0a^4+{}^4C_1a^3b+{}^4C_2a^2b^2+{}^4C_3ab^3+{}^4C_4b^4\right)\\ -\left({}^4C_0a^4-{}^4C_1a^3b+{}^4C_2a^2b^2-{}^4C_3ab^3+{}^4C_4b^4\right)\end{array}\right]$$
$$=2\left({}^4C_1a^3b+{}^4C_3ab^3\right)$$
$$=2\left(4a^3b+4ab^3\right)$$
$$=8ab\left(a^2+b^2\right)$$

Putting $a=\sqrt{3}$ and $b=\sqrt{2}$,

$$\left(\sqrt{3}+\sqrt{2}\right)^4-\left(\sqrt{3}-\sqrt{2}\right)^4=8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left[\left(\sqrt{3}\right)^2+\left(\sqrt{2}\right)^2\right]$$
$$=8\sqrt{6}[3+2]$$
$$=40\sqrt{6}$$

Question 12:

Find $(x+1)^6+(x-1)^6$. Hence, evaluate $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$.

Solution:

Using Binomial Theorem, the expression $(x+1)^6$ and $(x-1)^6$ can be expanded as

$$(x+1)^6={}^6C_0x^6+{}^6C_1x^5+{}^6C_2x^4+{}^6C_3x^3+{}^6C_4x^2+{}^6C_5x+{}^6C_6$$
$$(x-1)^6={}^6C_0x^6-{}^6C_1x^5+{}^6C_2x^4-{}^6C_3x^3+{}^6C_4x^2-{}^6C_5x+{}^6C_6$$

Therefore,

$$(x+1)^6+(x-1)^6=\left[\begin{array}{r}\left({}^6C_0x^6+{}^6C_1x^5+{}^6C_2x^4+{}^6C_3x^3+{}^6C_4x^2+{}^6C_5x+{}^6C_6\right)\\ +\left({}^6C_0x^6-{}^6C_1x^5+{}^6C_2x^4-{}^6C_3x^3+{}^6C_4x^2-{}^6C_5x+{}^6C_6\right)\end{array}\right]$$
$$=2\left[{}^6C_0x^6+{}^6C_2x^4+{}^6C_4x^2+{}^6C_6\right]$$
$$=2\left[x^6+15x^4+15x^2+1\right]$$

Putting $x=\sqrt{2}$

$$\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6=2\left[\left(\sqrt{2}\right)^6+15\left(\sqrt{2}\right)^4+15\left(\sqrt{2}\right)^2+1\right]$$
$$=2[8+15\times4+15\times2+1]$$
$$=2[8+60+30+1]$$
$$=2\times99$$
$$=198$$

Question 13:

Show that $9^{n+1}-8n-9$ is divisible by 64, whenever n is a positive integer.

Solution:

To show that $9^{n+1}-8n-9$ is divisible by 64, we need to prove that $9^{n+1}-8n-9=64k$ where k is some natural number.

By binomial theorem,

$$(1+a)^m = {}^mC_0 + {}^mC_1 a + {}^mC_2 a^2 + \ldots + {}^mC_m a^m$$

For $a=8$ and $m=n+1$, we obtain

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$$

Hence,

$$9^{n+1} = 1+(n+1)(8)+(8)^2\left[{}^{n+1}C_2 + {}^{n+1}C_3(8) + \ldots\ldots + {}^{n+1}C_{n+1}(8)^{n+1}\right]$$

$$9^{n+1} = 9+8n+64\left[{}^{n+1}C_2 + {}^{n+1}C_3(8) + \ldots\ldots + {}^{n+1}C_{n+1}(8)^{n+1}\right]$$

Therefore, $9^{n+1}-8n-9=64k$

Where $k=\left[{}^{n+1}C_2 + {}^{n+1}C_3(8) + \ldots\ldots + {}^{n+1}C_{n+1}(8)^{n+1}\right]$ is a natural number.

Thus, $9^{n+1}-8n-9$ is divisible by 64, whenever n is a positive integer.

Question 14:

Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$

Solution:

By Binomial theorem,

$$\sum_{r=0}^{n} {}^nC_r a^{n-r} b^r = (a+b)^n$$

By putting $b=3$ and $a=1$, we obtain

$$\sum_{r=0}^{n} {}^nC_r (1)^{n-r} 3^r = (1+3)^n$$

$$\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Hence Proved

EXERCISE 8.2

Question 1:

Find the coefficient of x^5 in $(x+3)^8$.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) in the binomial expression of $(a+b)^n$ is given by
$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming x^5 occurs in the expansion of $(x+3)^8$, we obtain

$$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 in (T_{r+1}), we obtain

$$8-r=5$$
$$r=3$$

Thus, the coefficient of x^5 is ${}^8C_3 (3)^3$

$$\begin{aligned} {}^8C_3 (3)^3 &= \frac{8!}{3!(5)!} \times (3)^3 \\ &= \frac{8 \times 7 \times 6 \times (5!)}{3 \times (2!) \times (5!)} \times 27 \\ &= 1512 \end{aligned}$$

Question 2:

Find the coefficient of $a^5 b^7$ in $(a-2b)^{12}$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) in the binomial expression of $(a+b)^n$ is given by
$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming $a^5 b^7$ occurs in the expansion of $(a-2b)^{12}$, we obtain

$$\begin{aligned} T_{r+1} &= {}^{12}C_r (a)^{12-r} (-2b)^r \\ &= {}^{12}C_r (-2)^r (a)^{12-r} (b)^r \end{aligned}$$

Comparing the indices of a and b in $a^5 b^7$ in (T_{r+1}), we obtain

$$r=7$$

Thus, the coefficient of $a^5 b^7$ is ${}^{12}C_7 (-2)^7$

$$
\begin{aligned}
{}^{12}C_7(-2)^7 &= \frac{12!}{7!(5)!}\times(-2)^7 \\
&= \frac{12\times11\times10\times9\times8\times(7!)}{(7!)\times(5!)}\times(-128) \\
&= -(792)\times(128) \\
&= -101376
\end{aligned}
$$

Question 3:

Write the general term in the expansion of $(x^2-y)^6$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) in the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Thus, the general term in the expansion of $(x^2-y)^6$ is

$$
\begin{aligned}
T_{r+1} &= {}^6C_r\left(x^2\right)^{6-4}(-y)^r \\
&= (-1)^r\,{}^6C_r(x)^{12-2r}(y)^r
\end{aligned}
$$

Question 4:

Write the general term in expansion of $(x^2-yx)^{12}, x \neq 0$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) in the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Thus, the general term in the expansion of $(x^2-yx)^{12}$ is

$$
\begin{aligned}
T_{r+1} &= {}^{12}C_r\left(x^2\right)^{12-r}(-yx)^r \\
&= {}^{12}C_r(x)^{24-2r}(-1)^r(y)^r(x)^r \\
&= (-1)^r\,{}^{12}C_r(x)^{24-r}(y)^r
\end{aligned}
$$

Question 5:

Find the 4^{th} term in the expansion of $(x-2y)^{12}$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) in the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Thus, the 4^{th} term in the expansion of $(x-2y)^{12}$ is

$$\begin{aligned} T_4 &= T_{3+1} \\ &= {}^{12}C_3 (x)^{12-3}(-2y)^3 \\ &= \frac{12!}{3!9!}(x)^9(-2)^3(y)^3 \\ &= \frac{12\times11\times10}{3\times2}\times(-8)x^9y^3 \\ &= -1760x^9y^3 \end{aligned}$$

Question 6:

Find the 13^{th} term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Thus, the 13^{th} term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$

$$\begin{aligned} T_{13} &= T_{12+1} \\ &= {}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12} \\ &= \frac{18!}{(12!)(6!)}\times9^6\times(x)^6\left(-\frac{1}{3}\right)^{12}\left(\frac{1}{\sqrt{x}}\right)^{12} \\ &= \frac{18\times17\times16\times15\times14\times13\times(12!)}{(12!)\times6\times5\times4\times3\times2}\times(3^{12})\times\left(\frac{1}{3^{12}}\right)\times(x^6)\times\left(\frac{1}{x^6}\right) \qquad \left[\because 9^6=(3^2)^6=3^{12}\right] \\ &= 18564 \end{aligned}$$

Question 7:

Find the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$

Solution:

It is known that in the expansion of $(a+b)^n$, when n is odd; there are two middle terms, namely $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion $\left(3-\frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th}=4^{th}$ term and $\left(\frac{7+1}{2}+1\right)^{th}=5^{th}$ term

$$
\begin{aligned}
T_4 &= T_{3+1}\\
&= {}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3\\
&= \frac{(7!)}{(3!)(4!)}\times(3^4)\times(-1)^3\times\left(\frac{x^9}{6^3}\right)\\
&= -\frac{7\times6\times5\times(4!)}{3\times2\times(4!)}\times(3^4)\times\left(\frac{1}{2^3\times3^3}\right)\times(x^9)\\
&= -\frac{105}{8}x^9
\end{aligned}
$$

Now,

$$
\begin{aligned}
T_5 &= T_{4+1}\\
&= {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4\\
&= \frac{7!}{(4!)(3)!}\times(3)^3\times\left(\frac{x^{12}}{6^4}\right)\\
&= \frac{7\cdot6\cdot5\cdot4!}{3\cdot2!(4)!}\times(3^3)\times\left(\frac{1}{2^4\times3^4}\right)\times(x^{12})\\
&= \frac{35}{48}x^{12}
\end{aligned}
$$

Thus, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$

Question 8:

Find the middle terms in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$

Solution:

It is known that in the expansion of $(a+b)^n$, when n is even; the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th}=6^{th}$ term

$$\begin{aligned}
T_6 &= T_{5+1} \\
&= {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 \\
&= \frac{(10!)}{(5!)(5!)}\times\left(\frac{x^5}{3^5}\right)\times\left(9^5\right)\times\left(y^5\right) \\
&= \frac{10\times9\times8\times7\times6\times(5!)}{5\times4\times3\times2\times(5!)}\times\left(\frac{1}{3^5}\right)\times\left(3^{10}\right)\times\left(x^5\right)\left(y^5\right) \\
&= 252\times3^5\times x^5\times y^5 \\
&= 61236x^5y^5
\end{aligned}$$

Question 9:

In the expansion of $(1+a)^{m+n}$, prove that the coefficients of a^m and a^n are equal.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) the binomial expression of $(a+b)^n$ is given by $T_{r+1}={}^nC_r a^{n-r}b^r$

Assuming that a^m occurs in the $(r+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{r+1}={}^{m+n}C_r(1)^{m+n-r}(a)^r={}^{m+n}C_r a^r$$

Comparing the indices of a in a^m and in T_{r+1}, we obtain $r=m$

Therefore, the coefficient of a^m is

$$\begin{aligned}
{}^{m+n}C_m &= \frac{(m+n)!}{m!(m+n-m)!} \\
&= \frac{(m+n)!}{(m!)(n!)} \qquad \ldots(1)
\end{aligned}$$

Assuming that a^n occurs in the $(k+1)^{th}$ term of the expansion $(1+a)^{m+n}$, we obtain

$$T_{k+1}={}^{m+n}C_k(1)^{m+n-k}(a)^k={}^{m+n}C_k a^k$$

Comparing the indices of a in a^n and in T_{k+1}, we obtain $k=n$

Therefore, the coefficient of a^n is

$$ {}^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} $$
$$ = \frac{(m+n)!}{(m!)(n!)} \qquad ...(2) $$

Thus from (1) and (2), it is clear that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal

Hence proved.

Question 10:

The coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ term in the expansion of $(x+1)^n$ are in the ratio of $1:3:5$. Find n and r.

Solution:

It is known that $(k+1)^{th}$ term, (T_{k+1}) term of the expansion $(a+b)^n$ is given by
$T_{k+1} = {}^nC_k a^{n-k} b^k$

Hence, $(r-1)^{th}$ term in the expansion of $(x+1)^n$ is

$$ T_{r-1} = {}^nC_{r-2}(x)^{n-(r-2)}(1)^{(r-2)} $$
$$ = {}^nC_{r-2}x^{n-r+2} $$

$(r+1)^{th}$ term in the expansion of $(x+1)^n$ is

$$ T_{r+1} = {}^nC_r(x)^{n-r}(1)^r $$
$$ = {}^nC_r x^{n-r} $$

r^{th} term in the expansion of $(x+1)^n$ is

$$ T_r = {}^nC_{r-1}(x)^{n-(r-1)}(1)^{(r-1)} $$
$$ = {}^nC_{r-1}x^{n-r+1} $$

Therefore, coefficients of the $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ term in the expansion of $(x+1)^n$ are ${}^nC_{r-2}$, ${}^nC_{r-1}$ and ${}^nC_r$ respectively.

Since these coefficients are in the ratio of $1:3:5$, we obtain

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!}$$

$$\frac{1}{3} = \frac{(r-1).(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!(n-r+1)!}$$

$$\frac{1}{3} = \frac{r-1}{n-r+2}$$

$$n-r+2 = 3r-3$$

$$n-4r+5 = 0$$

$$n = 4r-5 \quad \ldots(1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!}$$

$$\frac{3}{5} = \frac{r.(r-1)!(n-r)!}{(r-1)!(n-r+1)!(n-r)!}$$

$$\frac{3}{5} = \frac{r}{n-r+1}$$

$$3n-3r+3 = 5r$$

$$3n-8r+3 = 0 \quad \ldots(2)$$

From (1) and (2), we obtain

$$3(4r-5)-8r+3 = 0$$

$$12r-15-8r+3 = 0$$

$$4r-12 = 0$$

$$r = 3$$

Putting the value of r in (1), we obtain n

$$n = 4\times 3-5$$

$$= 12-5$$

$$= 7$$

Thus, $n=7$ and $r=3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1-x)^{2n-1}$.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that x^n occurs in the $(r+1)^{th}$ term of the expansion $(1+x)^{2n}$, we obtain

$$\begin{aligned} T_{r+1} &= {}^{2n}C_r (1)^{2n-r} (x)^r \\ &= {}^{2n}C_r x^r \end{aligned}$$

Comparing the indices of x in x^n and in T_{r+1}, we obtain $r = n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$

$$\begin{aligned} {}^{2n}C_n &= \frac{(2n)!}{n!(2n-n)!} \\ &= \frac{(2n)!}{(n!)(n!)} \\ &= \frac{(2n)!}{(n!)^2} \qquad \ldots(1) \end{aligned}$$

Assuming that x^n occurs in the $(k+1)^{th}$ term of the expansion $(1+x)^{2n-1}$, we obtain

$$\begin{aligned} T_{k+1} &= {}^{2n}C_k (1)^{2n-1-k} (x)^k \\ &= {}^{2n}C_k x^k \end{aligned}$$

Comparing the indices of x in x^n and in T_{k+1}, we obtain $k = n$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n-1}$

$$\begin{aligned}
{}^{2n-1}C_n &= \frac{(2n-1)!}{n!(2n-1-n)!} \\
&= \frac{(2n-1)!}{n!(n-1)!} \\
&= \frac{2n\cdot(2n-1)!}{2n\cdot n!(n-1)!} \\
&= \frac{(2n)!}{2\cdot(n!)(n!)} \\
&= \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \qquad \ldots(2)
\end{aligned}$$

From (1) and (2), we can observe that

$$\begin{aligned}
{}^{2n-1}C_n &= \frac{1}{2}\left({}^{2n}C_n\right) \\
{}^{2n}C_n &= 2\left({}^{2n-1}C_n\right)
\end{aligned}$$

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1-x)^{2n-1}$.
Hence Proved

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that x^2 occurs in the $(r+1)^{th}$ term of the expansion $(1+x)^m$, we obtain

$$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$$

Comparing the indices of x in x^2 in (T_{r+1}), we obtain $r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$.

$$
\begin{aligned}
{}^{m}C_2 &= 6 \\
\frac{m!}{2!(m-2)!} &= 6 \\
\frac{m(m-1)\times(m-2)!}{2\times(m-2)!} &= 6 \\
m(m-1) &= 12 \\
m^2 - m - 12 &= 0 \\
m^2 - 4m + 3m - 12 &= 0 \\
m(m-4) + 3(m-4) &= 0 \\
(m-4)(m+3) &= 0
\end{aligned}
$$

$m = 4$ or $m = -3$

Thus, 4 is the positive value of *m* for which the coefficient of x^2 in the expansion $(1+x)^m$ is 6.

MISCELLANEOUS EXERCISE

Question 1:

Find a, b and n in the expansion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290 and 30375 respectively.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) the binomial expression of $(a+b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$

The first three terms of expansion are 729, 7290 and 30375 respectively.

$$\begin{aligned} T_1 &= {}^nC_0 a^{n-0} b^0 \\ &= a^n \\ &= 729 \qquad \ldots(1) \end{aligned}$$

$$\begin{aligned} T_2 &= {}^nC_1 a^{n-1} b^1 \\ &= na^{n-1} b \\ &= 7290 \qquad \ldots(2) \end{aligned}$$

$$\begin{aligned} T_3 &= {}^nC_1 a^{n-2} b^2 \\ &= \frac{n(n-1)}{2} a^{n-2} b^2 \\ &= 30375 \qquad \ldots(3) \end{aligned}$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\frac{nb}{a} = 10 \qquad \ldots(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$
$$\frac{(n-1)b}{2a} = \frac{30375}{7290}$$
$$\frac{(n-1)b}{a} = \frac{30375\times 2}{7290}$$
$$\frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$10 - \frac{b}{a} = \frac{25}{3} \qquad \ldots(\text{from } 4)$$
$$\frac{b}{a} = 10 - \frac{25}{3}$$
$$\frac{b}{a} = \frac{5}{3} \qquad \ldots(5)$$

Putting $\frac{b}{a} = \frac{5}{3}$ in (4), we obtain

$$n \times \frac{5}{3} = 10$$
$$n = 6$$

Substituting $n = 6$ in (1), we obtain

$$a^6 = 729$$
$$a = \sqrt[6]{729}$$
$$= 3$$

From (5), we obtain
$$\frac{b}{3} = \frac{5}{3}$$
$$b = 5$$

Therefore, $a = 3, b = 5, n = 6$

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal.

Solution:

It is known that $(r+1)^{th}$ term, (T_{r+1}) the binomial expression of $(a+b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that x^2 occurs in the $(r+1)^{th}$ term of the expansion $(3+ax)^9$, we obtain

$$\begin{aligned} T_{r+1} &= {}^9C_r (3)^{9-r} (ax)^r \\ &= {}^9C_r (3)^{9-r} a^r x^r \end{aligned}$$

Comparing the indices of x in x^2 and in (T_{r+1}), we obtain $r = 2$

Thus, the coefficient of x^2 is

$$\begin{aligned} {}^9C_2 (3)^{9-2} a^2 &= \frac{9!}{2!7!}(3)^7 a^2 \\ &= 36(3)^7 a^2 \end{aligned}$$

Assuming that x^3 occurs in the $(k+1)^{th}$ term of the expansion $(3+ax)^9$, we obtain

$$\begin{aligned} T_{k+1} &= {}^9C_k (3)^{9-k} (ax)^k \\ &= {}^9C_k (3)^{9-k} a^k x^k \end{aligned}$$

Comparing the indices in x^2 and x^3 in (T_{r+1}), we obtain $k = 3$

Thus, the coefficient of x^3 is

$$\begin{aligned} {}^9C_3 (3)^{9-3} a^3 &= \frac{9!}{3!6!}(3)^6 a^3 \\ &= 84(3)^6 a^3 \end{aligned}$$

It is given that the coefficient of x^2 and x^3 are equal.

$$\begin{aligned} 84(3)^6 a^3 &= 36(3)^7 a^2 \\ 84a &= 36 \times 3 \\ a &= \frac{36 \times 3}{84} \\ &= \frac{9}{7} \end{aligned}$$

Hence, the required value of $a = \frac{9}{7}$

Question 3:

Find the coefficient of x^5 in the product $(1+2x)^6 (1-x)^7$ using binomial theorem.

Solution:

Using binomial theorem, $(1+2x)^6$ and $(1-x)^7$ can be expanded as

$$(1+2x)^6 = {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 + {}^6C_5(2x)^5 + {}^6C_6(2x)^6$$
$$= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6$$
$$= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$$

$$(1-x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 - {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7$$
$$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$$

Therefore,

$$(1+2x)^6(1-x)^7 = (1+12x+60x^2+160x^3+240x^4+192x^5+64x^6)(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7)$$
$$= 1(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1)$$
$$= 171x^5$$

Thus, the coefficient of x^5 in the product $(1+2x)^6(1-x)^7$ is 171.

Question 4:

If a and b are distinct integers, prove that $a-b$ is a factor of $a^n - b^n$ whenever n is a positive integer.

[Hint: write $a^n = (a-b+b)^n$ and expand]

Solution:

In order to prove that $a-b$ is a factor of $a^n - b^n$, we have to prove that $a^n - b^n = k(a-b)$ where k is some natural number.

We can write a as

$$a = a - b + b$$
$$a^n = (a-b+b)^n$$
$$a^n = [(a-b)+b]^n$$
$$a^n = {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_nb^n$$
$$a^n = (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n$$
$$a^n - b^n = (a-b)\left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$$
$$a^n - b^n = k(a-b)$$

where $k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$

This shows that $a-b$ is a factor of a^n-b^n whenever n is a positive integer.
Hence proved.

Question 5:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$

Solution:

Using binomial theorem,

$$\begin{aligned}(a+b)^6 &= {}^6C_0a^6+{}^6C_1a^5b+{}^6C_2a^4b^2+{}^6C_3a^3b^3+{}^6C_4a^2b^4+{}^6C_5ab^5+{}^6C_6b^6\\ &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\end{aligned}$$

$$\begin{aligned}(a-b)^6 &= {}^6C_0a^6-{}^6C_1a^5b+{}^6C_2a^4b^2-{}^6C_3a^3b^3+{}^6C_4a^2b^4-{}^6C_5ab^5+{}^6C_6b^6\\ &= a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\end{aligned}$$

Therefore,

$$(a+b)^6-(a-b)^6=2\left[6a^5b+20a^3b^3+6ab^5\right]$$

Putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain,

$$\begin{aligned}\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6 &= 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)+20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]\\ &= 2\left[54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right]\\ &= 2\times198\sqrt{6}\\ &= 396\sqrt{6}\end{aligned}$$

Question 6:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$

Solution:

Using binomial theorem,

$$\begin{aligned}(x+y)^4 &= {}^4C_0x^4+{}^4C_1x^3y+{}^4C_2x^2y^2+{}^4C_3xy^3+{}^4C_4y^4\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

$$\begin{aligned}(x-y)^4 &= {}^4C_0x^4-{}^4C_1x^3y+{}^4C_2x^2y^2-{}^4C_3xy^3+{}^4C_4y^4\\ &= x^4-4x^3y+6x^2y^2-4xy^3+y^4\end{aligned}$$

Therefore,

$$(x+y)^4+(x-y)^4=2\left(x^4+6x^2y^2+y^4\right)$$

Putting $x=a^2$ and $y=\sqrt{a^2-1}$, we obtain

$$\begin{aligned}\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4&=2\left[\left(a^2\right)^4+6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2+\left(\sqrt{a^2-1}\right)^4\right]\\&=2\left[a^8+6a^4\left(a^2-1\right)+\left(a^2-1\right)^2\right]\\&=2\left[a^8+6a^6-6a^4+a^4-2a^2+1\right]\\&=2\left[a^8+6a^6-5a^4-2a^2+1\right]\\&=2a^8+12a^6-10a^4-4a^2+2\end{aligned}$$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution:

We can express 0.99 as the difference of two numbers whose powers are easier to calculate.

Hence, $0.99=1-0.01$

Therefore,

$$\begin{aligned}(0.99)^5&=(1-0.01)^5\\&={}^5C_0(1)^5-{}^5C_1(1)^4(0.01)+{}^5C_2(1)^3(0.01)^2\\&=1-5(0.01)+10(0.01)^2\\&=1-0.05+0.001\\&=1.001-0.05\\&=0.951\end{aligned}$$

Hence, the approximation of $(0.99)^5$ is 0.951

Question 8:

Find *n*, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$

Solution:

In the expansion,

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots\ldots\ldots\ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$$

Fifth term from the beginning is ${}^nC_4a^{n-4}b^4$

Fifth term from the end is ${}^nC_4a^4b^{n-4}$

Therefore, in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$;

fifth term from the beginning is ${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and

fifth term from the end is ${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$$\begin{aligned}{}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 &= {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3}\\ &= \frac{n!}{6\cdot4!(n-4)!}\left(\sqrt[4]{2}\right)^n \qquad \ldots(1)\end{aligned}$$

$$\begin{aligned}{}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} &= {}^nC_{n-4}\frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n}\\ &= {}^nC_{n-4}\cdot2\cdot\frac{3}{\left(\sqrt[4]{3}\right)^n}\\ &= \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} \qquad \ldots(2)\end{aligned}$$

It is given that the ratio of of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$

Therefore,

$$\left[\frac{n!}{6\cdot 4!(n-4)!}\left(\sqrt[4]{2}\right)^n\right]:\left[\frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n}\right]=\sqrt{6}:1$$

$$\frac{\left(\sqrt[4]{2}\right)^n}{6}:\frac{6}{\left(\sqrt[4]{3}\right)^n}=\sqrt{6}:1$$

$$\frac{\left(\sqrt[4]{2}\right)^n}{6}\times\frac{\left(\sqrt[4]{3}\right)^n}{6}=\sqrt{6}$$

$$\left(\sqrt[4]{6}\right)^n=36\sqrt{6}$$

$$(6)^{\frac{n}{4}}=(6)^{\frac{5}{2}}$$

$$\frac{n}{4}=\frac{5}{2}$$

$$n=4\times\frac{5}{2}$$

$$=10$$

Thus the value of $n=10$.

Question 9:

Expand using binomial theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x\neq 0$

Solution:

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4={}^nC_0\left(1+\frac{x}{2}\right)^4-{}^nC_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)+{}^nC_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2-{}^nC_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3+{}^nC_4\left(\frac{2}{x}\right)^4$$

$$=\left(1+\frac{x}{2}\right)^4-4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)+6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right)-4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right)+\frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{24}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}-\frac{16}{x^2}+\frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4} \qquad \ldots(1)$$

Again, by using binomial theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$= 1 + 4\times\frac{x}{2} + 6\times\frac{x^4}{4} + 4\times\frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \qquad \ldots(2)$$

And

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \qquad \ldots(3)$$

From (1), (2) and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4 = 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

Question 10:

Find the expansion of $\left(3x^2 - 2ax + 3a^2\right)^3$

Solution:

Using binomial theorem, the expansion of $\left(3x^2 - 2ax + 3a^2\right)^3$ is given by

$$\left[\left(3x^2-2ax\right)+3a^2\right]^3 = {}^3C_0\left(3x^2-2ax\right)^3 + {}^3C_1\left(3x^2-2ax\right)^2\left(3a^2\right) + {}^3C_2\left(3x^2-2ax\right)\left(3a^2\right)^2 + {}^3C_3\left(3a^2\right)^3$$

$$= \left(3x^2-2ax\right)^3 + 3\left(9x^4 - 12ax^3 + 4a^2x^2\right)\left(3a^2\right) + 3\left(3x^2-2ax\right)\left(9a^4\right) + 27a^6$$

$$= \left(3x^2-2ax\right)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$$

$$= \left(3x^2-2ax\right)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \qquad \ldots(1)$$

Again, by using binomial theorem, we obtain

$$
\begin{aligned}
\left(3x^2-2ax\right)^3 &= {}^3C_0\left(3x^2\right)^3-{}^3C_1\left(3x^2\right)^2\left(2ax\right)+{}^3C_2\left(3x^2\right)\left(2ax\right)^2+{}^3C_3\left(2ax\right)^3 \\
&= 27x^6-3\left(9x^4\right)\left(2ax\right)+3\left(3x^2\right)\left(4a^2x^2\right)-8a^3x^3 \\
&= 27x^6-54ax^5+36a^2x^4-8a^3x^3 \qquad \ldots(2)
\end{aligned}
$$

From (1) and (2), we obtain

$$
\begin{aligned}
\left(3x^2-2ax+3a^2\right)^3 &= 27x^6-54ax^5+36a^2x^4-8a^3x^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6 \\
&= 27x^6-54ax^5+117a^2x^4-116a^3x^3+117a^4x^2-54a^5x+27a^6
\end{aligned}
$$